Boxicity and Maximum degree

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Abstract

An axis-parallel d–dimensional box is a Cartesian product $R_1 \times R_2 \times \cdots \times R_d$ where R_i (for $1 \leq i \leq d$) is a closed interval of the form $[a_i, b_i]$ on the real line. For a graph G, its *boxicity box(G)* is the minimum dimension d, such that G is representable as the intersection graph of (axis–parallel) boxes in d–dimensional space. The concept of boxicity finds applications in various areas such as ecology, operation research etc.

We show that for any graph G with maximum degree Δ, $box(G) \leq 2\Delta^2 + 2$. That the bound does not depend on the number of vertices is a bit surprising considering the fact that there are highly connected bounded degree graphs such as expander graphs. Our proof is very short and constructive. We conjecture that $box(G)$ is $O(\Delta)$.

Let $\mathcal{F} = \{S_x \subseteq U : x \in V\}$ be a family of subsets of a universe U, where V is an index set. The intersection graph $\Omega(\mathcal{F})$ of $\mathcal{F}$ has V as vertex set, and two distinct vertices x and y are adjacent if and only if $S_x \cap S_y \neq \emptyset$.

Representation of graphs as the intersection graphs of various geometrical objects is a well studied topic in graph theory. A prime example of a graph class defined in this way is the class of interval graphs: *A graph G is an interval graph* if and only if *G has an interval realization: i.e., each vertex of G can be associated to an interval on the real line such that two intervals intersect if and only if the corresponding vertices are adjacent.* Motivated by theoretical as well as practical considerations, graph theorists have tried to generalize the concept of interval graphs in various ways. One such generalization is the concept of *boxicity* defined as follows.

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interval graphs.

The concept of boxicity was introduced by F. S. Roberts [10]. It finds applications in niche overlap (competition) in ecology and to problems of fleet maintenance in operations research. (See [6].) It was shown by Cozzens [5] that computing the boxicity of a graph is NP–hard. This was later strengthened by Yannakakis [15], and finally by Kratochvil [9] who showed that deciding whether boxicity of a graph is at most 2 itself is NP–complete.

There have been many attempts to estimate or bound the boxicity of graph classes with special structure. In his pioneering work, F. S. Roberts proved that the boxicity of complete k–partite graphs are k. Scheinerman [11] showed that the boxicity of outer planar graphs is at most 2. Thomassen [13] proved that the boxicity of planar graphs is bounded above by 3. The boxicity of split graphs is investigated by Cozzens and Roberts [6]. In a recent manuscript [2] the authors showed that $box(G) \leq tw(G) + 2$ where $tw(G)$ is the treewidth of G. Little is known about the structure imposed on a graph by its high boxicity.

A number of NP-hard problems are known to be polynomial time solvable for interval graphs. Since boxicity is a direct generalization of the notion of interval graphs, such results may generalize to bounded boxicity graphs. Thus our result may be of interest from an algorithmic point of view.

Researchers have also tried to generalize or extend the concept of boxicity in various ways. The poset boxicity [14], the rectangular number [4], grid dimension [1], circular dimension [8,12] and the boxicity of digraphs [3] are some examples.

Let G be a simple, finite, undirected, unweighted graph on n vertices. Let $V(G)$ denote the vertex set of G and $E(G)$ denote the edge set of G. Let $\Delta(G)$ denote the maximum degree of G. Let $I_1, \ldots, I_k$ be k interval graphs such that $V(I_j) = V(G)$ for $1 \leq j \leq k$. If $E(G) = E(I_1) \cap \cdots \cap E(I_k)$, then we say that $I_1, \ldots, I_k$ is an *interval graph representation* of G. The following equivalence is well-known.

Fact [Roberts [10]] *The minimum k such that there exists an interval graph representation of G using k interval graphs $I_1, \ldots, I_k$ is the same as $box(G)$.*

Theorem 1 *For any graph G with maximum degree Δ, $box(G) \leq 2\Delta^2 + 2$.*

PROOF. Let $V(G) = V$. Let G^2 denote the *square* of G defined as follows: $V(G^2) = V$ and two vertices u and v are adjacent in G^2 if and only if the shortest distance between u and v in G is either 1 or 2. Let $\chi(G^2) = k$, where $\chi(H)$ denote the *chromatic number* of the graph H.

Consider an optimal vertex coloring $c : V \to \{1, \ldots, k\}$ of G^2. For $1 \leq i \leq k$, let $V_i = \{u \in V \mid c(u) = i\}$ be the ith color class. For $1 \leq i \leq k$, let G_i be defined

as follows. $V(G_i) = V$ and $E(G_i) = E(G) \cup \{(u,v) : u,v \in V - V_i$ and $u \neq v\}$. We claim that $E(G) = E(G_1) \cap \cdots \cap E(G_k)$. To see this, first observe that for $1 \leq i \leq k$, $E(G) \subseteq E(G_i)$. Now consider $(u,v) \notin E(G)$. Let $c(u) = i$. Then by construction, $(u,v) \notin E(G_i)$.

We now show that $box(G_i) \leq 2$ for $1 \leq i \leq k$. First we claim that in G, for any vertex $w \in V - V_i$, w has at most one neighbor in V_i. This is because, if w has two neighbors say x and y in V_i, then clearly x is adjacent to y in G^2 and thus they can not belong to the same color class V_i. Now, by construction of G_i, we have for any $(u,v) \in V_i \times (V - V_i)$, (u,v) belongs to $E(G_i)$ if and only if (u,v) belongs to $E(G)$. Thus it follows that with respect to G_i also, for any vertex $w \in V - V_i$, w has at most one neighbor in V_i. Without loss of generality, let the vertices in V_i be $\{1, \ldots, h\}$ where $h = |V_i|$. Consider two orderings π_0 and π_1 of V_i such that for any $j \in V_i$, $\pi_0(j) = j$ and $\pi_1(j) = h - j + 1$. For $r \in \{0,1\}$, define the interval graph I_r on the vertex set V as follows: For $w \in V_i$, let the interval $[\pi_r(w), \pi_r(w)]$ be assigned to w. For $w \in V - V_i$, if w has no neighbors in V_i with respect to G_i then assign the interval $[0,0]$ to w. Otherwise let z be its only neighbor in V_i with respect to G_i. Assign the interval $[0, \pi_r(z)]$ to w. We claim that $E(G_i) = E(I_0) \cap E(I_1)$ and thus $box(G_i) \leq 2$. By construction, it is clear that $E(G_i) \subseteq E(I_r)$ for $r \in \{0,1\}$. It remains to show that if $(u,v) \notin E(G_i)$ then either $(u,v) \notin E(I_0)$ or $(u,v) \notin E(I_1)$. Since $V - V_i$ induces a complete graph in G_i, if $(u,v) \notin E(G_i)$ then either $u,v \in V_i$ or $u \in V_i$ and $v \in V - V_i$. Clearly, by the construction, V_i is an independent set in I_0 as well as I_1. Thus the only case we have to consider is when $u \in V_i$ and $v \in V - V_i$. If v has no neighbors in V_i then clearly $(u,v) \notin E(I_0)$ and $(u,v) \notin E(I_1)$. Otherwise let x be the (only) neighbor of v in V_i. Now, clearly either $\pi_0(u) > \pi_0(x)$ or $\pi_1(u) > \pi_1(x)$. It follows that $(u,v) \notin E(I_0)$ or $(u,v) \notin E(I_1)$.

Recalling that $E(G) = \bigcap_{i=1}^{k} E(G_i)$, it follows that $box(G) \leq \sum_{i=1}^{k} box(G_i) \leq 2k$. Now using the well-known fact that for any graph H, $\chi(H) \leq \Delta(H) + 1$, (see chapter 5, [7]) it follows that $k \leq \Delta(G^2) + 1 \leq \Delta^2 + 1$ and the result follows. ∎

Remark. We conjecture that $box(G)$ is $O(\Delta)$. In fact, given any Δ, it is not difficult to construct graphs of boxicity $\Omega(\Delta)$ on arbitrarily large number of vertices, using a construction given by Roberts [10].

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